Sec. 13.2:
Derivatives and Integrals of Vector Functions

What We Will Go Over In Section 13.2

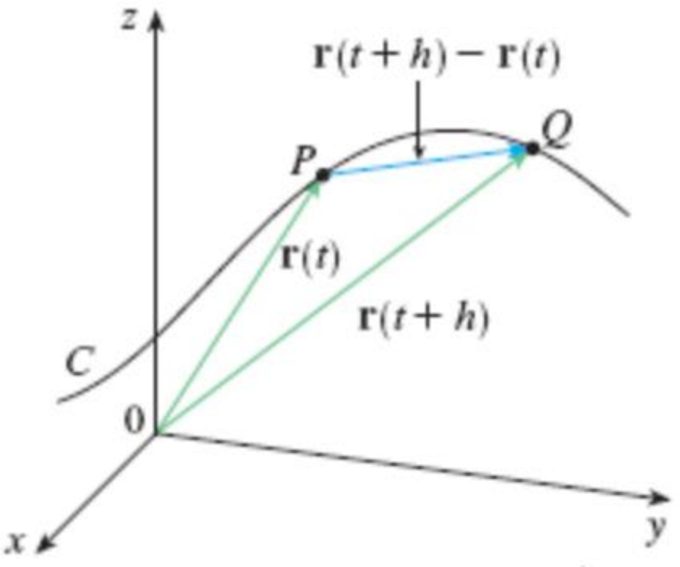
1. Derivatives of Vector Functions
2. Differentiation Rules
3. Integrals

1. Derivatives of Vector Functions

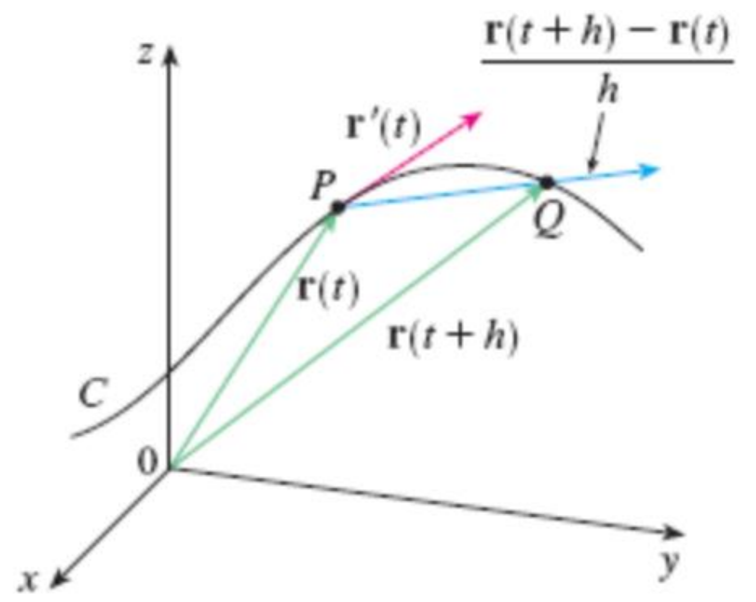
<u>Definition</u>: Given a vector function $\vec{r}(t)$, it <u>derivative</u> is the vector function

$$\frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

for all values of t for which this limit exists.



(a) The secant vector $\overrightarrow{PQ}$



(b) The tangent vector $\mathbf{r}'(t)$

1. Derivatives of Vector Functions

<u>Definition</u>: Given a vector function $\vec{r}(t)$, it <u>derivative</u> is the vector function

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

for all values of t for which this limit exists.

<u>Notes</u>:

- $\vec{r}\,'(t)$ is called the tangent vector to the curve at P
- The tangent line to C at P is defined to be the line through P parallel to the tangent vector $\vec{r}\,'(t)$
- The unit tangent vector is defined as…

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

1. Derivatives of Vector Functions

<u>Theorem 2</u>: To find the derivative of a vector function, just differentiate each component.

2 Theorem

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

1. Derivatives of Vector Functions

Ex 1: Let $\vec{r}(t) = < \ tan^{-1}t\,, 2\mathrm{e}^{2t}\,,\ 8te^{t}\ >$.

a) Find the derivative of $\vec{r}(t)$

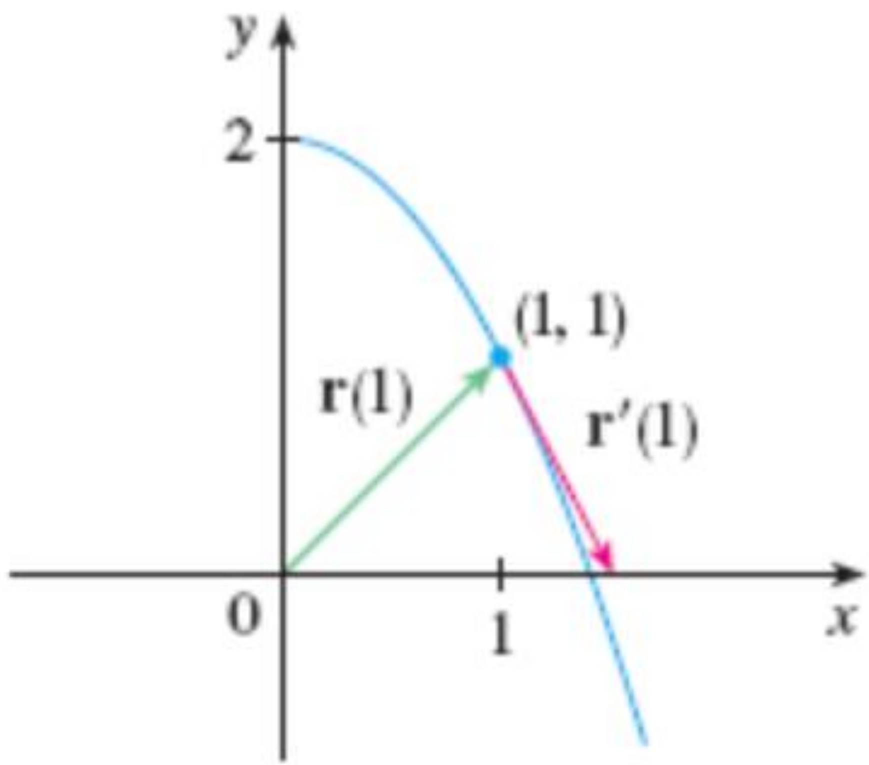
b) Find the unit tangent vector $\vec{T}(t)$ at the point where $t = 0$

1. Derivatives of Vector Functions

<u>Ex 2</u>: For the curve $\vec{r}(t) = < \sqrt{t}\,,\ 2 - t\ >$, find $\vec{r}'(t)$ and sketch the position vector $\vec{r}(1)$ and the tangent vector $\vec{r}'(1)$.

1. Derivatives of Vector Functions

Ex 2: For the curve $\vec{r}(t) = < \sqrt{t}\,,\ 2 - t\ >$, find $\vec{r}'(t)$ and sketch the position vector $\vec{r}(1)$ and the tangent vector $\vec{r}'(1)$.



1. Derivatives of Vector Functions

Ex 3: Find parametric equations for the tangent line to the curve given below at the specified point.

$$x = \sqrt{t^2 + 3}\,, \quad y = \ln\left(t^2 + 3\right), \quad z = t \ ; \qquad (2, \ln 4, 1)$$

1. Derivatives of Vector Functions

Definition: The second derivative of a vector function is just the derivative of the derivative of the function. That is…

$$r'' = (r')'$$

Ex 4: For the vector function $\vec{r}(t) = < \mathrm{t}^4 + 2\mathrm{t},\ \cos \mathrm{t},\ \mathrm{e}^{\mathrm{t}} >$, find $r''(\mathrm{t})$.

2. Differentiation Rules

3 Theorem

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\dfrac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$
4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$ (Chain Rule)

2. Differentiation Rules

Ex 5: Show that if $|\vec{r}(t)| = c$ (a constant) , then r'(t) is orthogonal to $\vec{r}(t)$ for all t.

(Geometrically, this means that for any curve on a sphere with center origin, the tangent vector is always perpendicular to the radius of the sphere).

3. Integrals

We can also calculate definite and indefinite integrals of a vector function.

Indefinite Integral

To find an indefinite integral, just find an antiderivative of each component and don't forget to add the constant of integration (which is a vector).

3. Integrals

We can also calculate definite and indefinite integrals of a vector function.

Definite Integral

There are 2 ways to find a definite integral…

1) Find the definite integral of each component.

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

3. Integrals

We can also calculate definite and indefinite integrals of a vector function.

Definite Integral

There are 2 ways to find a definite integral…

2) Use the fundamental theorem of calculus for vector functions…
   - Find any antiderivative of the vector function
   - Plug in the top number then the bottom number to the antiderivative and subtract the results

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

3. Integrals

Ex 6: Find…

a) $\int \sec^2 t\,\vec{i} + t\left(t^2+1\right)^3\vec{j} + t^2\ln t\,\vec{k}\;\;dt$

3. Integrals

<u>Ex 6</u>: Find…

b) $\int_0^2 < t, -t^3, 3t^5 > \; dt$ (in 2 ways)